

# Controllable Subsets in Graphs

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October 18, 2010

## Abstract

Let  $X$  be a graph on  $v$  vertices with adjacency matrix  $A$ , and let  $S$  be a subset of its vertices with characteristic vector  $z$ . We say that the pair  $(X, S)$  is *controllable* if the vectors  $A^r z$  for  $r = 1, \dots, v-1$  span  $\mathbb{R}^v$ . Our concern is chiefly with the cases where  $S = V(X)$ , or  $S$  is a single vertex. In this paper we develop the basic theory of controllable pairs. We will see that if  $(X, S)$  is controllable then the only automorphism of  $X$  that fixes  $S$  as a set is the identity. If  $(X, S)$  is controllable for some subset  $S$  then the eigenvalues of  $A$  are all simple.

## 1 Introduction

Let  $X$  be a graph on  $v$  vertices with adjacency matrix  $A$ . If  $z \in \mathbb{R}^v$ , define the matrix  $W$  by

$$W = (z \ A z \ \dots \ A^{v-1} z)$$

The pair  $(A, z)$  is *controllable* if  $W$  is invertible. In this article,  $z$  will often be the characteristic vector of some subset  $S$  of  $V(X)$ , and then we will say that  $(X, S)$  is controllable if  $(A, z)$  is. When  $S \subseteq V(X)$ , the entries of  $W = W_S$  counts walks in the graph  $X$ , and we will call  $W_S$  the *walk matrix* of  $S$ .

We note one interesting property of controllable pairs.

**1.1 Lemma.** *If  $(X, S)$  is controllable, then any automorphism of  $X$  that fixes  $S$  as a set is the identity.*

*Proof.* We view the automorphisms of  $X$  as permutation matrices that commute with  $A$ . Let  $z$  be the characteristic vector of  $S$ . An automorphism of  $X$  fixes  $S$  if and only if  $Pz = z$ . If  $Pz = z$  then for  $r = 0, \dots, v-1$

$$PA^r z = A^r Pz = A^r z$$

and therefore  $PW_S = W_S$ . Hence if  $W_S$  is invertible,  $P = I$ .  $\square$

In this paper we develop the theory of controllable pairs. We will see that there is a close connection to the subject of control theory. The ideas in this paper have already been put to use in quantum physics—see [1].

## 2 Characterizations of Controllability

We derive some useful characterizations of controllability.

Assume that  $A = A(X)$  has the spectral decomposition

$$A = \sum_{\theta} \theta E_{\theta}.$$

Then

$$A^r z = \sum_{\theta} \theta^r E_{\theta} z$$

and hence  $\text{col}(W)$  is spanned by the vectors  $E_{\theta} z$ , that is, by the nonzero projections of  $z$  onto the distinct eigenspaces of  $A$ . We say that an eigenvalue  $\theta$  is in the *support* of  $y$  if  $E_{\theta} y \neq 0$ . Equivalently,  $\theta$  is in the support of  $y$  if  $y^T E_{\theta} y \neq 0$ . The *dual degree* of  $y$  is  $|\text{supp}(y)| - 1$ . So  $(X, S)$  is controllable if and only if the dual degree of  $S$  is  $v - 1$ . Note that if  $X$  is connected and  $z$  is the eigenvector of  $X$  with eigenvalue equal to the spectral radius of  $A$ , then by the Perron-Frobenius theory, all entries of  $z$  are positive. It follows that the dual degree is non-negative.

From the spectral decomposition of  $A$ , we see that

$$(tI - A)^{-1} z = \sum_{\theta} \frac{1}{t - \theta} E_{\theta} z$$

and hence

$$z^T (tI - A)^{-1} z = \sum_{\theta} \frac{z^T E_{\theta} z}{t - \theta}$$

Since

$$z^T E_\theta z = z^T E_\theta^2 z = (E\theta z)^T E_\theta x \quad (1)$$

we have that  $z^T E_\theta z = 0$  if and only if  $E_\theta z = 0$ . Therefore the rank of  $W_S$  is equal to the number of poles of the rational function  $z^T(tI - A)^{-1}z$ . There is a polynomial  $\phi_S(X, t)$  with degree at most  $v - 1$  such that

$$z^T(tI - A)^{-1}z = \frac{\phi_S(X, t)}{\phi(X, t)}$$

(It is not hard to show that, if  $S$  is the vertex  $u$ , then  $\phi_S(X, t) = \phi(X \setminus u, t)$ .)

This provides a useful characterization of controllability:

**2.1 Lemma.** *Let  $X$  be a graph on  $n$  vertices and suppose  $S \subseteq V(X)$ . Let  $z$  be the characteristic vector of  $S$ . Then  $(X, S)$  is controllable if and only if the rational function  $z^T(tI - A)^{-1}z$  has  $n$  distinct poles.  $\square$*

Our next result characterizes controllability in terms of linear algebra rather than rational functions,

**2.2 Theorem.** *Let  $S$  be a subset of the vertices of the graph  $X$ , with characteristic vector  $z$ . The following statements are equivalent:*

- (a)  $(X, S)$  is controllable.
- (b) The matrices  $A$  and  $zz^T$  generate the algebra of all  $v \times v$  matrices.
- (c) The matrices  $A^i z z^T A^j$  where  $0 \leq i, j < v$  form a basis for the algebra of all  $v \times v$  matrices.

*Proof.* We show that (a) and (c) are equivalent.

If  $(X, S)$  is controllable, then the vectors

$$z, Az, \dots, A^{v-1}z$$

are linearly independent in  $\mathbb{R}^v$ . Since

$$\mathbb{R}^v \otimes \mathbb{R}^v \cong \text{Mat}_{v \times v}(\mathbb{R})$$

it follows that the matrices

$$(A^i z)(A^j z)^T = A^i z z^T A^j, \quad (0 \leq i, j < v)$$

are linearly independent in  $\text{Mat}_{v \times v}(\mathbb{R})$ . On the other hand, if  $(X, S)$  is not controllable, then the vectors  $A^r z$  span a space of dimension at most  $v - 1$ , and the matrices  $A^i z z^T A^j$  span a space of dimension at most  $(v - 1)^2$ .

To complete the proof, note that

$$z z^T A^r z z^T = (z^T A^r z) z z^T$$

and therefore any element of the algebra generated by  $A$  and  $z z^T$  is a linear combination of matrices of the form

$$A^r z z^T A^s, \quad (r, s \geq 0)$$

Since  $A$  is  $v \times v$ , any polynomial in  $A$  is a linear combination of the powers

$$I, A, \dots, A^{v-1},$$

We conclude that (b) implies (c). Since (b) is an immediate consequence of (c), we are done.  $\square$

### 3 Isomorphism

In this section we consider pairs that need not be controllable.

Let  $X$  be a graph on  $v$  vertices with adjacency matrix  $A$  and let  $y$  be a vector in  $\mathbb{R}^v$ . Let  $Y$  be a graph on  $v$  vertices with adjacency matrix  $B$  and let  $z$  be a vector in  $\mathbb{R}^v$ . We say that the pairs  $(X, y)$  and  $(Y, z)$  are *isomorphic* if there is an orthogonal matrix  $L$  such that

$$L A L^T = B, \quad L y = z.$$

In this case  $L W_y = W_z$ ; thus controllability is preserved by isomorphism. Further

$$W_z^T W_z = W_y^T L^T L W_y = W_y^T W_y.$$

We will occasionally refer to the characteristic polynomial of the matrix  $A$  as the characteristic polynomial of the pair  $(A, z)$ .

If the pairs  $(A, y)$  and  $(B, z)$  are isomorphic, then  $A$  and  $B$  must have the same characteristic polynomial.

**3.1 Theorem.** *Two pairs  $(A, y)$  and  $(B, z)$  are isomorphic if and only if  $A$  and  $B$  are similar and*

$$y^T (I - tA)^{-1} y = z^T (I - tB)^{-1} z.$$

*Proof.* We have seen that the necessity of this condition is an easy consequence of the definition. So we assume that  $A$  and  $B$  are cospectral and that our two rational functions are equal. From our remarks at the start of Section 2, in particular (1), the latter condition implies that  $y$  and  $z$  have the same support and that  $y^T E_\theta y = z^T F_\theta z$  for each eigenvalue  $\theta$  in  $\text{supp}(y)$ .

We construct two orthonormal bases for  $\mathbb{R}^v$ ; the linear map that takes the first basis to the second will be our isomorphism.

Let

$$A = \sum_{\theta} \theta E_\theta, \quad B = \sum_{\theta} \theta F_\theta$$

be the spectral decompositions of  $A$  and  $B$ .

Construct an orthonormal basis  $\alpha$  for  $\mathbb{R}^v$  as follows. Suppose  $\text{rk}(W_y) = d$ . The first  $d$  vectors of the basis will be the normalizations of the non-zero vectors  $E_\theta y$ . For each  $\theta$  in  $\text{supp}(y)$ , add an orthonormal basis for the subspace of eigenvectors for  $A$  with eigenvalue  $\theta$  that are orthogonal to  $E_\theta y$ . If  $\theta \notin \text{supp}(y)$ , add an orthonormal basis for the  $\theta$ -eigenspace of  $A$ . By the same procedure we can form an orthonormal basis  $\beta$  relative to  $B$  and  $y$  and, possibly after some rearrangement, we may assume that the vectors  $\alpha_i$  and  $\beta_i$  have the same eigenvalue for each  $i$ . If  $L$  is the matrix representing the unique linear mapping that sends  $\alpha$  to  $\beta$ , then  $L$  is orthogonal and  $LAL^T = B$ .

Set  $W = W_y$  and let  $Y$  be the matrix whose columns are the nonzero vectors  $E_\theta y$ . Then  $W$  and  $Y$  have the same column space, in fact if  $M$  is the  $d \times v$  matrix whose  $ij$ -entry is  $\theta^{j-1}$ , where  $\theta$  is the  $i$ -th eigenvalue in  $\text{supp}(y)$ , then  $W = YV$ . Note that  $V$  is determined by the support of  $y$ . The columns of  $Y$  are pairwise orthogonal and therefore

$$W^T W = V^T D V,$$

where  $D$  is the diagonal matrix with diagonal entries of the form  $y^T E_\theta y$ . Since  $y^T (I - tA)^{-1} y = z^T (I - tB)^{-1} z$ , we infer that  $W_y^T W_y = W_z^T W_z$  and, since  $V$  has a right inverse, we conclude that

$$y^T E_\theta y = z^T F_\theta z.$$

for each eigenvalue  $\theta$  in  $\text{supp}(y)$ . Since

$$y = \sum_{\theta \in \text{supp}(y)} E_\theta y$$

and since  $L$  maps  $(y^T E_\theta y)^{-1/2} E_\theta y$  to  $(z^T F_\theta z)^{-1/2} F_\theta z$ , it follows that  $Ly = z$ .  $\square$

Note that the two rational functions above are equal if and only if

$$y^T E_\theta y = z^T F_\theta z$$

for **all** eigenvalues  $\theta$ .

**3.2 Corollary.** *If  $(A, y)$  and  $(B, y)$  are controllable and  $y^T(I - tA)^{-1}y = z^T(I - tB)^{-1}z$ , then  $(A, y)$  and  $(B, z)$  are isomorphic.*

*Proof.* If  $(A, y)$  is controllable, then the eigenvalues of  $A$  are distinct and each one is a pole of  $y^T(I - tA)^{-1}y$ . So our hypothesis implies that  $A$  and  $B$  are cospectral and that  $y^T E_\theta y = z^T F_\theta z$  for all eigenvalues  $\theta$ .  $\square$

By Lemma 2.2 in [2] (for example) it follows that if  $X$  and  $Y$  are cospectral then  $\overline{X}$  and  $\overline{Y}$  are cospectral if and only if

$$\mathbf{1}^T(I - tA(X))^{-1}\mathbf{1} = \mathbf{1}^T(I - tA(Y))^{-1}\mathbf{1}.$$

So the results of this section imply the important result of Johnson and Newman [3] that if  $X$  and  $Y$  are cospectral with cospectral complements, then there is an orthogonal matrix  $L$  such that

$$L^T A(X)L = A(Y), \quad L^T(A(\overline{X}))L = A(\overline{Y}).$$

## 4 Graph Theory

If  $S \subseteq V(X)$ , we define the *covering radius* of  $S$  to be the least integer  $r$  such that each vertex of  $X$  is at distance at most  $r$  from a vertex of  $S$ . Thus  $S$  has covering radius equal to 1 if and only if it is a dominating set, and the diameter of  $X$  is the maximum value of the covering radius of a vertex.

**4.1 Lemma.** *If  $S$  has dual degree  $m$  and covering radius  $r$ , then  $r \leq m$ .*

*Proof.* If  $v \in V(X)$ , then  $(A^i z)_v$  is equal to the number of walks of length  $i$  from  $v$  to a vertex in  $S$ . It follows that  $((A + I)^i z)_v$  is zero if and only if  $i$  is less than  $\text{dist}(v, S)$ . From this it follows in turn that the vectors

$$z, Az, \dots, A^r z$$

are linearly independent and therefore  $r + 1$  is a lower bound on  $\text{rk}(W_z)$ .  $\square$

One consequence of this lemma is the well known result that if  $X$  has diameter  $d$ , then  $d + 1$  is less than or equal to the number of distinct eigenvalues of  $A$ . As an example, if  $X$  is the path  $P_n$  on  $n$  vertices and  $S$  is one of its end-vertices, then covering radius of  $S$  is  $n - 1$ . Hence the dual degree of  $S$  is  $n - 1$ , from which we deduce the well known fact that the eigenvalues of the path are distinct.

**4.2 Lemma.** *If  $X$  is vertex transitive and  $|V(X)| > 2$ , no subset of  $V(X)$  is controllable.*

*Proof.* If  $X$  has a controllable subset with characteristic vector  $z$ , then the vectors  $E_\theta z$  form a basic for  $\mathbb{R}^v$ , and thus  $X$  has  $v$  simple eigenvalues. But the only vertex transitive graph with all eigenvalues simple is  $K_2$ .  $\square$

If  $S \subseteq V(X)$ , we define the *cone of  $X$  relative to  $S$*  to be the graph we get by taking one new vertex and declaring it to be adjacent to each vertex in  $S$ .

**4.3 Theorem.** *The pairs  $(X, S)$  and  $(Y, T)$  are isomorphic if and only if  $X$  is cospectral to  $Y$  and the cone of  $X$  relative to  $S$  is cospectral to the cone of  $Y$  relative to  $T$ .*

*Proof.* Let  $b$  denote the characteristic vector of  $S$  and let  $\widehat{X}$  denote the cone over  $X$  relative to  $S$ . Then

$$A(\widehat{X}) = \begin{pmatrix} 0 & b^T \\ b & A \end{pmatrix}$$

and so

$$\begin{pmatrix} t & -b^T \\ -b & tI - A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & tI - A \end{pmatrix} \begin{pmatrix} t & -b^T \\ -(tI - A)^{-1}b & I \end{pmatrix}.$$

Accordingly

$$\phi(\widehat{X}, t) = \phi(X, t)(t - b^T(tI - A)^{-1}b)$$

and this yields that

$$\frac{\phi(\widehat{X}, t)}{\phi(X, t)} = t - \sum_{\theta} \frac{b^T E_\theta b}{t - \theta}.$$

Our result follows now from Theorem 3.1.  $\square$

**4.4 Theorem.** Suppose  $V(X) = \{1, \dots, v\}$  and  $S \subseteq V(X)$ . Construct the cone  $\widehat{X}$  by joining the vertex 0 to each vertex in  $S$ . Then  $(X, S)$  is controllable if and only if  $(\widehat{X}, \{0\})$  is controllable.

*Proof.* Assume  $v = |V(X)|$ . If  $b$  is the characteristic vector of  $S$ , we have

$$\frac{\phi(\widehat{X}, t)}{\phi(X, t)} = t - \sum_{\theta} \frac{b^T E_{\theta} b}{t - \theta}. \quad (2)$$

Further  $(X, S)$  is controllable if and only if this rational function has  $v$  distinct poles.

Now

$$e_0^T (tI - \widehat{A})^{-1} e_0 = ((tI - \widehat{A})^{-1})_{0,0} = \frac{\phi(X, t)}{\phi(\widehat{X}, t)}$$

and therefore  $(\widehat{X}, \{0\})$  is controllable if and only if the rational function  $\phi(X, t)/\phi(\widehat{X}, t)$  has  $v+1$  distinct poles, that is, if and only if  $\phi(\widehat{X}, t)/\phi(X, t)$  has exactly  $v+1$  distinct zeros. Since the derivative of the right side in (2) is positive everywhere it is defined, between each pair of consecutive zeros there is exactly one pole. Therefore there are  $v+1$  distinct zeros.

The following corollary provides infinite families of controllable pairs.

**4.5 Corollary.** Let  $S$  be a subset of  $V(X)$ , and let  $Y_k$  be the graph obtained by taking a path on  $k$  vertices and joining one of its end-vertices to each vertex in  $S$ . Let 0 denote the other end-vertex of the path. If  $(X, S)$  is controllable then  $(Y_k, \{0\})$  is controllable.  $\square$

Our next result generalizes Lemma 2.4 from [5].

**4.6 Lemma.** Suppose the pairs  $(X, S)$  and  $(Y, T)$  are isomorphic and controllable. Then the matrix  $W_T W_S^{-1}$  represents the isomorphism from  $(X, S)$  to  $(Y, T)$ .

*Proof.* Let  $A$  and  $B$  be the adjacency matrices of  $X$  and  $Y$  respectively.

Since the pairs are isomorphic,  $W_S^T W_S = W_T^T W_T$ . Since they are controllable,  $W_S$  and  $W_T$  are invertible and therefore

$$W_T W_S^{-1} = W_T^{-T} W_S^T = (W_T W_S^{-1})^{-T}$$

Hence  $Q + W_T W_S^{-1}$  is orthogonal.

Let  $C$  denote the companion matrix of  $\phi(X, t)$ . Then

$$AW_S = W_S C$$

and, since  $A$  and  $B$  are similar,

$$BW_T = W_T C.$$

Hence

$$BW_T W_S^{-1} = W_T C W_S^{-1} = W_T W_S^{-1} A$$

and thus  $B = QAQ^{-1}$ .

Let  $y$  and  $z$  be the characteristic vectors of  $S$  and  $T$  respectively. Since  $QW_S = W_T$ , we certainly have  $Qy = z$ .  $\square$

**4.7 Corollary.** *If the pairs  $(X, S)$  and  $(X, T)$  are isomorphic and controllable and  $Q = W_T W_S^{-1}$ , then  $Q$  commutes with  $A(X)$  and  $Q^2 = I$ .*

*Proof.* From the lemma we have  $QAQ^{-1} = A$ , so  $Q$  and  $A$  commute. Since the eigenvalues of  $A$  are all simple, this implies that  $Q$  is a polynomial in  $A$  and therefore it is symmetric.  $\square$

When the hypotheses of this corollary hold, the matrix  $Q$  can be viewed as a kind of “approximate” automorphism of order two—it is rational, commutes with  $A$  and swaps the characteristic vectors of  $S$  and  $T$ . If  $S$  and  $T$  are single vertices  $u$  and  $v$ , then  $Q$  will be block diagonal with one block of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the other an orthogonal matrix of order  $(v-2) \times (v-2)$  which commutes with the adjacency matrix of  $X \setminus \{u, v\}$ .

## 5 Controllable Graphs

We say that graph is *controllable* if  $(X, V(X))$  is controllable. Since any automorphism of  $X$  fixes  $V(X)$ , we see that a controllable graph is asymmetric. We can see this another way. If  $(X, V(X))$  is controllable, then  $W$  is invertible and so if  $e_u^T W = e_v^T W$ , then  $u = v$ . One consequence of this observation is that the ordering of the vertices obtained from the lexicographic ordering

of the rows of  $W$  is canonical: two controllable graphs are isomorphic if and only their ordered walk matrices are equal.

It is also immediate that a graph is controllable if and only if its complement is.

**5.1 Theorem.** *If  $u$  and  $v$  are cospectral vertices in  $X$ , then the  $A$ -modules generated by  $e_u + e_v$  and  $e_u - e_v$  are orthogonal. If the  $A$ -module generated by  $\{e_u, e_v\}$  is  $\mathbb{R}^{V(X)}$ , then  $\mathbb{R}^{V(X)}$  is the direct sum of these two cyclic modules.*

*Proof.* If  $u$  and  $v$  are cospectral, then  $(E_\theta)_{u,u} = (E_\theta)_{v,v}$  for each eigenvalue  $\theta$  of  $X$ . For any projection  $E_\theta$  we have

$$(e_u + e_v)^T E_\theta (e_u - e_v) = (E_\theta)_{u,u} - (E_\theta)_{v,v}$$

and so the vectors  $E_\theta(e_u + e_v)$  are orthogonal to the vectors  $E_\tau(e_u - e_v)$ , for all choices of  $\theta$  and  $\tau$ .  $\square$

The second condition in the theorem will hold if  $u$  (or  $v$ ) is controllable. The theorem implies that, if  $u$  and  $v$  are cospectral and  $z$  lies in the  $A$ -module generated by  $e_u + e_v$ , then  $z_u = z_v$ .

We have the following consequence of Lemma 2.1 and the remark preceding it:

**5.2 Lemma.** *A vertex  $u$  in  $X$  is controllable if and only if  $\phi(X \setminus u, t)$  and  $\phi(X, t)$  are coprime.*  $\square$

For the path  $P_n$  on  $n$  vertices we have

$$\phi(P_0, t) = 1, \quad \phi(P_1, t) = t$$

and, if  $n \geq 1$ ,

$$\phi(P_{n+1}, t) = t\phi(P_n, t) - \phi(P_{n-1}, t)$$

from which it follows by induction that  $\phi(P_{n+1}, t)$  and  $\phi(P_n, t)$  are coprime for all  $n$ . So if 1 is an end-vertex of  $P_n$ , the pair  $(P_n, \{1\})$  is controllable.

**5.3 Corollary.** *If the characteristic polynomial of  $X$  is irreducible over the rationals, then  $(X, V(X))$  is controllable and  $(X, u)$  is controllable for any vertex  $u$ .*  $\square$

In [2] it is proved that controllable graphs are reconstructible. We conjecture that almost all graphs are controllable.

## 6 Laplacians

The theory we have presented will hold for any symmetric matrix. If  $D$  is the diagonal matrix of valencies of the vertices of  $X$ , then

$$L := D - A$$

is the *Laplacian* of  $X$ . This is a symmetric matrix with row sums zero. If  $e_i$  and  $e_j$  are two of the standard basis vectors, then

$$H_{i,j} := (e_i - e_j)(e_i - e_j)^T$$

If the graph  $Y$  is obtained by adding the edge  $ij$  to  $X$ , then

$$L(Y) = D - A + H_{i,j}.$$

Thus

$$L(X) = \sum_{ij \in E(X)} H_{i,j}.$$

Now

$$\begin{aligned} \det(tI - L - H_{i,j}) &= \det[(tI - L)(I - (tI - L)^{-1})H_{i,j}] \\ &= \det(tI - L) \det(I - (tI - L)^{-1}(e_i - e_j)(e_i - e_j)^T) \\ &= \det(tI - L)(1 - (e_i - e_j)^T(tI - L)^{-1}(e_i - e_j)) \end{aligned}$$

and if  $h := e_i - e_j$ , then

$$\frac{\phi(L(Y), t)}{\phi(L(X), t)} = 1 - h^T(tI - L)^{-1}h = 1 - \sum_{\lambda} \frac{h^T F_{\lambda} h}{t - \lambda}$$

where  $L = \sum_{\lambda} \lambda F_{\lambda}$  is the spectral decomposition of  $L$ . It follows that the eigenvalues of  $L(Y)$  are determined by the eigenvalues of  $L(X)$  along with the squared lengths of the projections of  $e_i - e_j$  onto the eigenspaces of  $L(X)$ .

If we get  $Y$  from  $X$  by deleting the edge  $ij$ , then we find that

$$\frac{\phi(L(Y), t)}{\phi(L(X), t)} = 1 + \sum_{\lambda} \frac{h^T F_{\lambda} h}{t - \lambda}$$

Let  $h$  denote  $e_i - e_j$ .

We observe that  $A^r h$  is orthogonal to  $\mathbf{1}$ , and so the dimension of the  $A$ -module generated by  $h$  is at most  $v - 1$ . We say that the pair of vertices  $\{i, j\}$  is *controllable* relative to the Laplacian if

$$(\mathbf{1} \quad h \quad Lh \quad \dots \quad L^{v-1}h)$$

has rank  $v - 1$ .

If  $ij$  is controllable and  $P$  is an automorphism of  $X$  that fixes  $\{i, j\}$ , then either

$$P(e_i - e_j) = e_i - e_j$$

and  $PW = W$ , or

$$P(e_i - e_j) = e_j - e_i$$

and  $PW = -W$ . In the latter case  $P = -I$  and so it is not a permutation matrix, in the former case  $P = I$ . We conclude that if  $ij$  is controllable, then only the identity automorphism fixes the set  $ij$ .

## 7 Control Theory

In this section we provide a brief introduction to some concepts from control theory. Our favorite source for this material is the book of Kailath [4] (but there is a lot of choice).

Consider a discrete system whose state at time  $n$  is  $x_n$ , where  $x_n \in \mathbb{F}^d$ . The states are related by the recurrence

$$x_{n+1} = Ax_n + u_n B \quad (n \geq 0). \quad (3)$$

where  $A$  and  $b$  are fixed matrices and the  $(u_n)_{n \geq 0}$  is arbitrary. The output  $c_n$  at time  $n$  is equal to  $c^T x_n$ , where  $c$  is fixed. The basic problem is determine information about the state of the system given  $(u_n)$  and  $(c_n)$ . From (3) we find that

$$\sum_{n \geq 0} t^n x_{n+1} = A \sum_{n \geq 0} t^n x_n + \left( \sum_{n \geq 0} u_n t^n \right) b$$

If we define

$$X(t) := \sum_{n \geq 0} t^n x_n, \quad u(t) := \sum_{n \geq 0} u_n t^n, \quad c(t) := \sum_{n \geq 0} c_n t^n$$

then we may rewrite our recurrence as

$$t^{-1}(X(t) - x_0) = AX(t) + u(t)b$$

and consequently

$$X(t) = (I - tA)^{-1}x_0 + tu(t)(I - tA)^{-1}b. \quad (4)$$

Thus we have two distinct contributions to the behavior of the system: one determined entirely by  $A$  and the initial state  $x_0$ , the other determined by  $A$ ,  $b$  and  $u(t)$ . It follows from (4) that the state of the system is always in the column space of the *controllability matrix*

$$W = (b \quad Ab \quad \dots \quad A^{d-1}b)$$

The system is *controllable* if  $W$  is invertible.

(Note that our “exposition” of control theory is confined to the simplest case. In general  $b$  and  $c$  are replaced by matrices  $B$  and  $C$ . The system is then controllable if the the  $A$ -module generated by  $\text{col}(B)$  is  $\mathbb{F}^v$ , and observable if the module generated by  $\text{col}(C)$  is  $\mathbb{R}^v$ . This more general case forced itself on us in our treatment of Laplacians.)

It is convenient to assume  $x_0 = 0$ . Then we have

$$c(t) = tu(t) c^T (I - tA)^{-1}b.$$

If the *observability matrix*

$$\begin{pmatrix} c^T \\ c^T A \\ \vdots \\ c^T A^{d-1} \end{pmatrix}$$

is invertible, then it is possible to infer the state of the system at time  $m$  from the observations  $c_m, \dots, c_{m+d-1}$ . In this case we say that the system is *observable*. Note that the system is observable if and only the pair  $(A, b)$  is controllable.

We can consider a more general version of (3): suppose  $A$  is  $n \times n$  and  $B$  is  $n \times k$ . We then have a system

$$x_{n+1} = Ax_n + Bu_n, \quad (n \geq 0)$$

where now  $u \in \mathbb{R}^k$ . In this case the system is controllable if the  $A$ -module generated by the column space of  $B$  is  $\mathbb{R}^n$ . This case arose in Theorem 5.1.

The series

$$c^T(I - tA)^{-1}b,$$

is known as the *transfer function* of the system. In control theory our variable  $t$  is normally replaced by a variable  $z^{-1}$ ; thus the transfer function becomes  $c^T(zI - A)^{-1}b$ .

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